Exponential Notation and Factoring Whole Numbers

On a clear night, thousands of stars are visible. Some stars appear larger and brighter than others. Their distances from Earth also vary greatly. For instance, the distance to the star Altair (see http://en.wikipedia.org/wiki/Altair) is about 100,000,000,000,000 miles from the Earth. These distances are soooo large that they are unmanageable as written whole numbers. It is easier to write very large numbers using exponents. The distance from Earth to Altair in words is (write it out in space provided)

and can be written in exponential form as 10^{14} miles. The exponent, 14, means to use the base, 10, as a factor 14 times.

$$10^{14} = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

10 is used as a factor 14 times
or
14 copies of 10 are multiplied together

$$=100,000,000,000,000$$

A whole number **exponent** indicates the number of times the **base** is used as a factor. An exponent is also called a **power**. Note that 10 can be written as 10^1. When there is no exponent, it is understood to be 1. A number such as 10^{14} is in **exponential form** and is read as "ten to the fourteenth power"

1. The distance from Earth to the Great Whirlpool Galaxy is 10^{20} miles. Write this distance as a whole number.

CELESTIAL BODY	DISTANCE FROM EARTH IN MILES	WRITTEN USING AN EXPONENT
Barnard's Galaxy, first known dwarf galaxy, discovered in 1882	10,000,000,000,000,000,000	10^{19}
Brightest quasar, 3C 273		10^{22}
Comet Hale-Bopp on April 6, 2091		10^{10}
Double star Shuart 1	1,000,000,000,000,000	10^{15}
First magnitude star, Altair	100,000,000,000,000	
First near-Earth asteroid, Eros, at its closest	10,000,000	
Great Whirlpool Galaxy	100,000,000,000,000,000,000	
Ionosphere		10^{2}
Mars on Nov. 2, 2001	100,000,000	
Million-star globular cluster Omega Centauri	100,000,000,000,000,000	
Nothing known about things at this distance		10^{12}
Russian *Molnyia* (Lightning) communications satellites at highest altitude		10^{4}
Saturn on Oct. 17, 2015	1,000,000,000	
Space shuttle when you lose sight of it	1,000	
Stratosphere	10	
Typical near-Earth asteroid when it flies by	1,000,000	

2. Referring to the chart to the left,
a. Write the distance from Earth to Barnard's Galaxy as a whole number.

b. Now write the distance to Barnard's Galaxy in base 10 using an exponent.

c. What is the relationship between the number of zeros that make up this whole number and the exponent when you write this number as a power?

3. Fill in the missing whole numbers and powers of 10 in the chart to the left

Example 1: A space shuttle is no longer visible to the naked eye once it is $1000 = 10^3$ miles away from you. Since 10 can be written as the product $2 \cdot 5$, you can rewrite 10^3 as $(2 \cdot 5)^3$. So it turns out that

$$1000 = 10^3 = (2 \cdot 5)^3 = (2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5) = 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 = 2^3 \cdot 5^3$$

4. Show how (explain why) $(3 \cdot 5)^2 = 3^2 \cdot 5^2$ using Example 1's method.

When a number is written as a product of its factors, it is called a **factorization** of the number. When all the factors are prime numbers, the product is called the **prime factorization** of the number. (Recall that a prime number is a whole number greater than 1 whose only whole-number factors are itself and 1)

Fundamental property of whole numbers: For any whole number, there is only one prime factorization.

5. a. $10 \cdot 10 \cdot 10 = 10^3$ is a factorization of 1000. Is 10^3 a prime factorization of 1000? Explain.

b. In Example 1, 10^3 was rewritten as $(2 \cdot 5)^3$. Is $(2 \cdot 5)^3$ a prime factorization of 1000?

c. If you ignore the order of the factors, how many prime factorizations of 1000 are there?

6. a. List the prime numbers that are less than 50.

b. What is the smallest prime number?

7. a. Determine the prime factorization of 90.

b. Describe a way to find the prime factorization of any whole number.

c. Use your method in part b to find the prime factorization of 100.

8. The distance to the double tar Stuart 1 is 1,000,000,000,000,000 (or 10^{15}) miles. Determine the prime factorization of this number. Write your result in both types of exponential forms (see problem 4).

9.The National Collegiate Athletic Association (NCAA) Men's Basketball Tournament is a major TV event each year. Sixty-four colleges are invited to participate in the tournament based on their seasonal records and performance in their conference playoffs. The teams are then paired and half of the teams are eliminated after each round of play. After round one, there are 32 teams, then 16, 8, 4, 2, and finally 1.

a. Determine the prime factorization of the numbers 64, 32, 16, 8, 4, and 2 and write each number using an exponent. The first entry, 64, is done for you.

NUMBER OF TEAMS	PRIME FACTORIZATION	WRITTEN USING AN EXPONENT
64	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^6
32		
16		
8		
4		
2		
Winner		

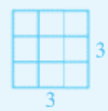
b. Describe the pattern for the exponents in the last column of this chart.

c. If you continue the pattern in the last column and write 1 in terms of the base, 2, what exponent would you attach to base 2

Any nonzero whole number raised to the zero power is equal to 1.

Example 2: *The Square of a Number*

In geometry, a square is a rectangle in which all the sides have the same length. The area of a square is a product of two factors, each equal to the length of a side. In exponential form, you say that the length is "squared".

A square with sides 1 unit in length has an area equal to 1 square unit.		$1^2 = 1$
A square with sides 2 units in length has an area equal to 4 square units.		$2^2 = 4$
A square with sides 3 units in length has an area equal to 9 square units.		$3^2 = 9$

10. a. Determine the area of a square whose sides are 5 units in length.

b. Determine the area of a square whose sides are 11 units in length.

11. Explain how to determine the square of any number.

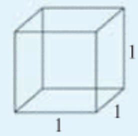
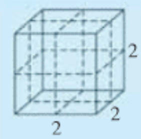
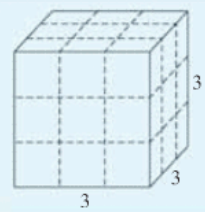
A whole number is **square** (sometimes called a *perfect square*) if it is the product of a whole number times itself. For example, 36 and 100 are both square numbers because $36 = 6^2$ and $100 = 10^2$

12. a. Draw a square that has an area of 49 square units on the grid.

b. What is the length of each side?

Example 3: *The Cube of a Number*

In geometry, a cube is a box in which all the edges have equal length. The volume (or space inside) of the cube is the product of three factors, each equal to the length of an edge. We say the volume is the length of the edge "cubed". The following pictures illustrate this idea.

A cube with edges 1 unit in length has a volume equal to 1 cubic unit.		$1^3 = 1$
A cube with edges 2 units in length has a volume equal to 8 cubic units.		$2^3 = 8$
A cube with edges 3 units in length has a volume equal to 27 cubic units.		$3^3 = 27$

13. a. Determine the volume of a cube whose edges are 4 units in length.

b. Determine the area of a square whose sides are 8 units in length.

14. Explain how to calculate the volume of a cube of any number.

15. You are a cake designer and have a client who is organizing a Monte Carlo Night for a charity benefit. The client wants several pairs of cakes that look like a pair of dice (cubes). You have 6-inch square pans. The square area of the bottom of the cake in each pan will be 36 square inches.

a. How high will a cake have to be to represent a die (cube)?

b. What will be the volume of the cake in cubic inches?

Perhaps you showed the calculation in Problem 15b as $6^2 \cdot 6^1 = 6 \cdot 6 \cdot 6 =$ 216 cubic inches
$= 216 \text{ in}^3$

Note that $6^2 \cdot 6^1 = 6^3$ and that the sum of the exponents is $2 + 1 = 3$.

When multiplying two numbers written in exponential form that have the same base, add the exponents. This sum becomes the new exponent attached to the original base. For example,

$$5^4 \cdot 5^3 = 5^7$$

since the product is the result of multiplying seven factors of 5.

$$(5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5) = 5^7$$

16. The Great Whirlpool Galaxy is one million time farther away from Earth than the first-magnitude star Altair. Use exponents to express this relationship (refer to chart in problem 2).

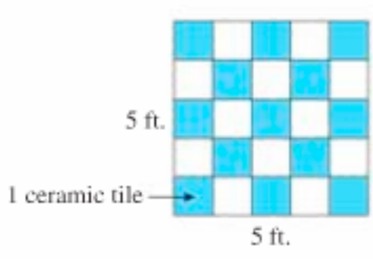
17. Review the following numbers using a single exponent. Check with your calculator.

a. $10^5 \cdot 10^4$ **b.** $4^7 \cdot 4^5$

Example 4: Cermanic Tile

Cermanic floor tiles can be square and measure 1 foot by 1 foot in size. If you want to tile a square space that is 5 feet by 5 feet as shown in the figure below, you would need 25 ceramic tiles. The 5-foot sides are called the **dimensions** of the 25 square foot area.

Numerically, you express the relationship between the dimensions and the area as $5 \cdot 5 = 5^2 = 25$. The factor, 5, appears twice in this factorization of 25, and is called the **square root** of 25. Using symbols, $\sqrt{25} = 5$, which is read as "the square root of 25 is 5"



18. Given that each tile has dimensions 1 inch by 1 inch:

a. Determine the dimensions of the square area that you could time with 81 ceramic tiles.

b. Determine the dimensions of the square area that you could time with 144 ceramic tiles.

19. **a.** Determine the square roots of 64 and 225.

b. Find the square root key on your calculator and check your answers in part a

20. **a.** Can you tile a square area with 100 of the 1-foot square tiles? Explain.

b. Can you tile a square area with 24 of the 1-foot square tiles? Explain.

Summary:

1. A whole-number **exponent** indicates the number of times to use the **base** as a factor. A number written as 10^{14} is in **exponential form**. The expression is called a power of 10.
2. Writing a number as a product of its factors is called **factorization**. When all the factors are prime numbers, the product is called the **prime factorization** of the number. A **prime number** is a whole number greater than 1 whose only whole number factors are itself and 1.
3. **Fundamental property of whole numbers:** For any whole number, there is only one factorization.
4. Any nonzero whole number raised to the **zero power** equals 1.
5. When multiplying number written in exponential form that have the same base, add the exponents. This sum is the new exponent on the original base. For example,
$$9^3 \cdot 9^4 = 9^7$$
6. A **square** is a rectangle in which all the sides have equal length. The area of a square is a product of two factors, each equal to the length of a side, that is the length squared.
7. A whole number is a perfect **square** if it can be rewritten as the product of two whole-number factors that are equal, that is, as the square of a whole number
8. The **square root** of a whole number is one of the two equal factors whose product is a whole number. For example, 6 is the square root of 36 because $6^2 = 36$. Using symbols, $\sqrt{36} = 6$.

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Exercise Set 2

1. The distance from Earth to Alnilam, the center star in Orion's belt, is 10^{16} miles. Write 10^{16} as a whole number.

2. The distance from Earth to the Large Magellan Cloud, a satellite galaxy of the Milky Way, is 10,000,000,000,000,000,000 miles. Write this distance in exponential form.

3. Determine two prime numbers between 50 and 60.

4. How many prime numbers are there between 60 and 70? List them.

5. List all possible factorizations of the following numbers.

a. 6

b. 15

c. 35

d. 22

6. How many factorizations do each of the last problem's numbers have? How many prime factorizations?

7. **a.** List all possible factorizations of 30 and 105.

b. How many different factorizations do each of these numbers have?

8. **a.** List all possible factorizations of 4, 9, and 25.

b. What do the factorizations of these three numbers share in common?

9. Determine the prime factorizations of each number:

a. 12

b. 75

c. 42

d. 96

10. Computers come with 1024, 512, 256, 128, or 64 MB of RAM (random access memory) Write the prime factorizations of 1024, 512, 256, 128, and 64 in exponential form. What do the factorizations share in common?

11. Write each exponential form as a whole number:

a. 3^0

b. 9^2

c. 5^4

d. 2^5

e. 12^2

12. Write each expression using a single exponent. Check your answers with a calculator.

a. $5^7 \cdot 5^8$

b. $9^2 \cdot 9^5$

c. $7^4 \cdot 7^7$

d. $5^{72} \cdot 5^0$

13. Determine if the given area can be the area of a square whose side has a whole number length.

a. 144 square feet (ft^2)
b. 160 ft^2
c. 664 ft^2
d. 256 ft^2