Evaluate each limit and then state what the limit tells you graphically. If the limit is an infinite limit, state the behaviour of the graph.

1. $\lim_{x \to 0^+} \frac{1}{x}$

2. $\lim_{x \to 0^-} \frac{1}{x}$

3. $\lim_{x \to \infty} \frac{1}{x}$

4. $\lim_{x \to -\infty} \frac{1}{x}$

5. $\lim_{x \to 0^+} \ln x$

6. $\lim_{x \to \infty} \ln x$

7. $\lim_{x \to \infty} e^x$

8. $\lim_{x \to -\infty} e^x$

9. $\lim_{x \to 0} \sin x$

10. $\lim_{x \to 0} \cos x$

11. $\lim_{x \to \pi/2} \sin x$

12. $\lim_{x \to \pi/2} \cos x$

13. $\lim_{x \to 1} (x-1)$

14. $\lim_{x \to 1} \left(\frac{x^2 - 1}{x - 1} \right)$

Indeterminate Forms

If both $f(x)\to 0$ and $g(x)\to 0$, as $x\to a$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

If this is a limit of a rational function, you can try to factor and then cancel out the common factor from the numerator and denominator, or rationalize the numerator, in order to evaluate the limit. If this doesn't work, then try a magical rule called *l'Hôpital's Rule* (described below).

If both $f(x)\to \pm\infty$ and $g(x)\to \pm\infty$, as $x\to a$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

If this is a limit of a rational function, you can try to factor and then cancel out the common factor from the numerator and denominator, or rationalize the numerator, in order to evaluate the limit. If this doesn't work, then try a magical rule called *l'Hôpital's Rule* (described below).

L'Hôpital's Rule

Suppose that f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contains a (except possibly at a).
Also, suppose that

$$\lim_{x\to a}\frac{f(x)}{g(x)} \text{ is an indeterminate form of the type } \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right exists (or is ∞ or $-\infty$).

Use l'Hôpital's Rule to find the following limit.

15. $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$

16. Prove that l'Hôpital's Rule is true for the case in which $f(a)=g(a)=0$, f' and g' are continuous and $g'(a)\neq 0$.

Find the limit and state what each limit means graphically. Use l'Hôpital's Rule *where appropriate*. If there is a more elementary method, consider using it. If l'Hôpital's Rule doesn't apply, explain why.

17. $\lim_{x\to 0} \frac{\sin x}{x}$

18. $\lim_{x\to 1} \frac{\ln x}{x-1}$

19. $\lim_{x\to \infty} \frac{\ln x}{x-1}$

20. $\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

21. $\lim_{\theta\to \pi/2} \frac{1-\sin\theta}{\csc\theta}$

22. $\lim_{x \to 0} \frac{x}{\tan^{-1}(4x)}$

23. $\lim_{x \to a^+} \frac{\cos x \ln(x-a)}{\ln(e^x - e^a)}$

24. $\lim_{x \to 0} \frac{x - \sin x}{x - \tan x}$

Indeterminate Products

If both $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$), as $x \to a$, then $\lim_{x \to a} f(x)g(x)$ is an indeterminate form of type $0 \cdot \infty$.

Try writing the product fg as $fg = \frac{f}{1/g}$ or as $fg = \frac{g}{1/f}$ in order to get the limit written in an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so that you can use l'Hôpital's Rule.

25. $\lim_{x \to -\infty} x^2 e^x$

26. $\lim_{x \to \infty} x \tan\left(\frac{1}{x}\right)$

27. $\lim_{x \to \pi/4} (1 - \tan x)\sec x$

Indeterminate Differences

If both $f(x) \to \infty$ and $g(x) \to \infty$, as $x \to a$, then $\lim_{x \to a}[f(x) - g(x)]$ is an indeterminate form of type $\infty - \infty$. Try writing the difference $f - g$, as a quotient by using a commond denominator or by factoring out a common factor, in order to get the limit written in an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so that you can use l'Hôpital's Rule.

Find the limit and state what each limit means graphically.

28. $\lim_{x \to 0}\left(\cot x - \frac{1}{x}\right)$

29. $\lim_{x \to \infty}\left(xe^{1/x} - x\right)$

Indeterminate Powers

There are different indeterminate forms that arise from $\lim_{x\to a}\left[f(x)\right]^{g(x)}$:

1. $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ gives you the indeterminate type 0^0
2. $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ gives you the indeterminate type ∞^0
3. $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\pm\infty$ gives you the indeterminate type 1^∞

To evaluate these types of limits, either create an equation $y=$ (the function) and use a natural logarithm on both sides of the equation, or write the function as an exponential $\lim_{x\to a}\left[f(x)\right]^{g(x)}=e^{g(x)\ln f(x)}$. After doing so you may or may not be able to use l'Hôpital's Rule.

Find the limit and state what each limit means graphically.

30. $\lim_{x\to\infty}\left(e^x+x\right)^{1/x}$

31. $\lim_{x \to 0^+} (\tan 2x)^x$