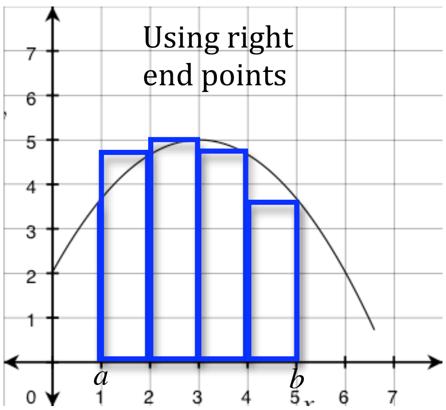
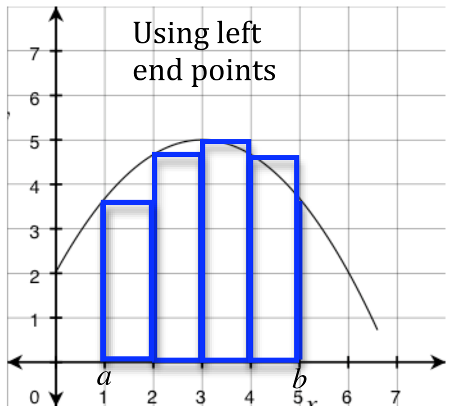
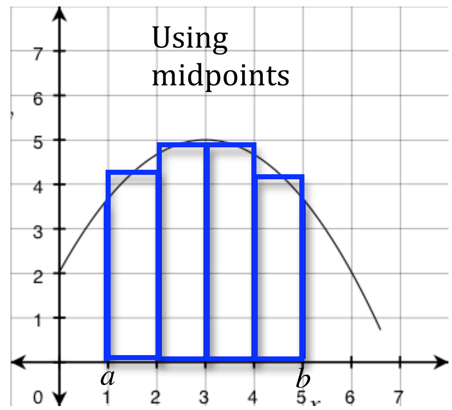
Chapters 2, 3, and 4 discussed differential calculus, which is based on derivatives. Derivatives were introduced using tangent and velocity problems. Chapter 5 begins the study of integral calculus. In Section 5.3, we will study how differential and integral calculus are related. First though, we'll start by looking at a basic area problem.

How could you estimate the area under the curve $y = f(x)$ for a to b, bounder by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$ and the x-axis ?

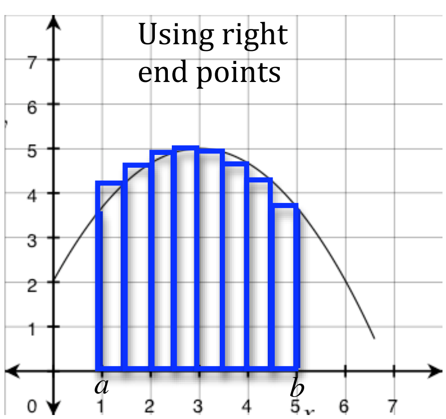
Split the interval $[a,b]$ into n=4 subintervals.
Use the specified points from each subinterval to determine the height of a rectangle over each subinterval.

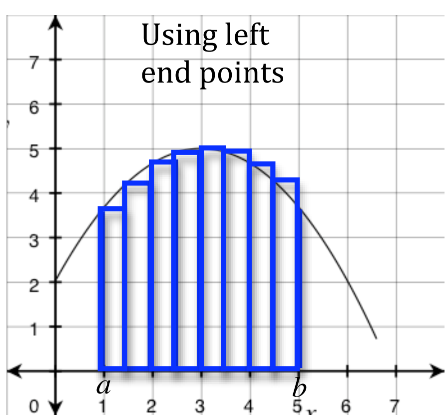


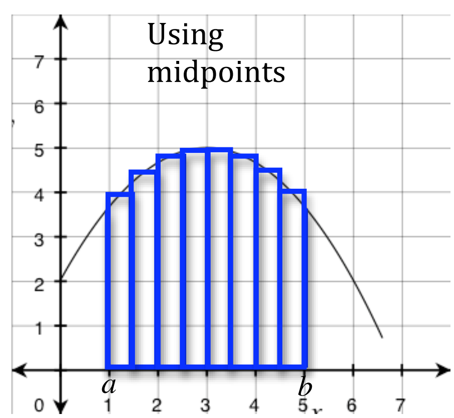




Split the interval $[a,b]$ into n=8 subintervals.
Use the specified points from each subinterval to determine the height of a rectangle over each subinterval.







Area under a Curve

The area A of the region S that lies under the curve of a **continuous** function f is the limit of the sum of approximating rectangle areas:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\underbrace{\overset{\text{(height)(Base)}}{f(x_1)\Delta x} + \overset{\text{(height)(Base)}}{f(x_2)\Delta x} + \cdots + \overset{\text{(height)(Base)}}{f(x_n)\Delta x}}\right]$$

Sum of a lot of Rectangle Areas. The bigger n gets, the more Rectangles there…and the more Rectangles there are, the better the sum approximates the actual area under the curve.

Let's look at a shorter way to write a sum of many terms (summation).

Summation (Sigma) Notation

If $a_m, a_{m+1}, a_{m+2}, \ldots, a_{n-1}, a_n$ are real numbers and m and n are integers such that $m \le n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n,$$

or in function notation:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

i is called the index of summation and it takes on consecutive integer values beginning with m and ending with n.

Write out each sum.

1. $\displaystyle\sum_{i=1}^{4} i^2$

2. $\displaystyle\sum_{i=0}^{n} 2^i$

3. $\displaystyle\sum_{k=1}^{4} \frac{k-1}{k}$

Write the sum in sigma notation. There is more than one correct answer for each sum.

4. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

5. $\dfrac{1}{1} + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \dfrac{1}{25} + \dfrac{1}{36}$

6. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

If F is an antiderivative of f on an interval I Then the most general antiderivative of f on I is $F(x)+C$ where C is an arbitrary constant.

1. $f(x)=\cos x$

2. $f(x)=\dfrac{1}{x}$

3. $f(x)=x^n$

4. $f(x)=\dfrac{1}{\sqrt{1-x^2}}$

Summation Properties

If c is any constant (that is, it does not depend on i) then

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i \qquad \sum_{i=m}^{n}(a_i+b_i)=\sum_{i=m}^{n}a_i+\sum_{i=m}^{n}b_i \qquad \sum_{i=m}^{n}(a_i-b_i)=\sum_{i=m}^{n}a_i-\sum_{i=m}^{n}b_i$$

7. Prove $\displaystyle\sum_{i=1}^{n} i = 1+2+3+4+\cdots+n=\frac{n(n+1)}{2}$

8. Use induction to prove $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Find the value of the sum.

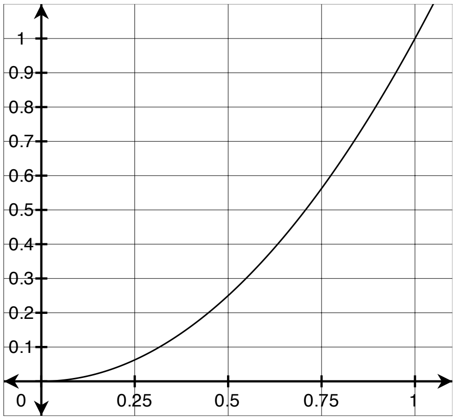
9. $\sum_{i=3}^{6} i(i+2)$

10. $\sum_{i=1}^{100} 4$

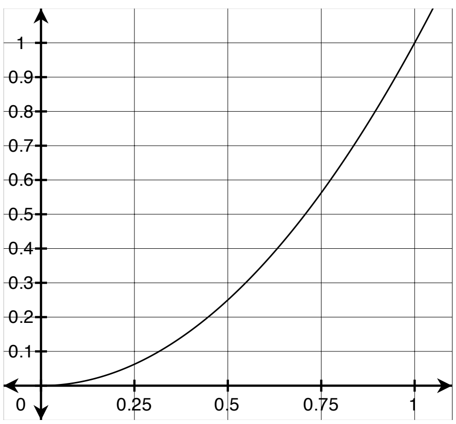
11. $\sum_{i=1}^{n} (3+2i)^2$

12. Estimate the area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 1$

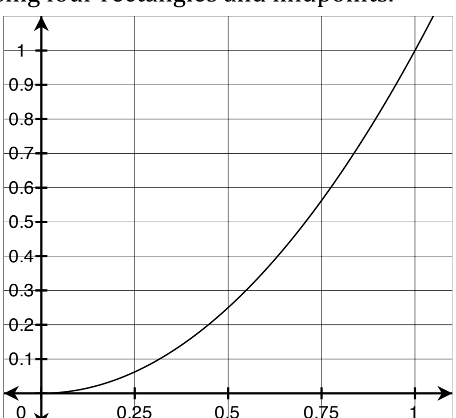
a) Using four rectangles and right endpoints



b) Using four rectangles and left endpoints.



c) Using four rectangles and midpoints.



d) Now find the exact area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 1$, by finding

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

12 (cont.)

The area under a curve can represent many different quantities. In the next example, it represents the distance a car has traveled over a certain time interval.

13. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.